CREST and 4th Ritsumeikan-Florence Workshop on Risk Simulation and Related Topics

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Lectures on Credit Risk

1. Models for single default
2. Contagion models
3. **Credit derivatives**

Some remarks

Assume that there exists a traded asset with price S, an $\mathbb{F}$-adapted process, and that the market is arbitrage free, using $\mathbb{G}$ adapted strategies. Assume furthermore than the interest rate if $\mathbb{F}$ adapted. Then there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$ such that $Se^{\int_0^t r_s ds}$ is a $\mathbb{G}$ martingale, hence an $\mathbb{F}$ martingale. If the market where S is traded is complete, immersion property holds true under $\mathbb{Q}$ and

$$\mathbb{Q}(\tau > t|\mathcal{F}_t) = \mathbb{Q}(\tau > t|\mathcal{F}_\infty)$$

If the market where S is traded is incomplete, immersion property holds true under some $\mathbb{Q}$.

If we are dealing only with defaultable asset, one can work in a general setting

In that case, a quite general assumption is the existence of a density under $\mathbb{P}$, i.e.,

$$\mathbb{P}(\tau > \theta|\mathcal{F}_t) = \int_\theta^\infty g_t(u)\nu(du)$$

where ν has no atoms.

Then,

- Immersion property is equivalent to $g_t(u) = g_u(u)$ for $t > u$
- $G_t = m_t - \int_0^t p_s(s)\nu(ds)$ where m is an $\mathbb{F}$-martingale
- $H_t - \int_0^t \frac{p_s(s)}{G_s}\nu(ds)$ is a $\mathbb{G}$ martingale
- Y is a $\mathbb{G}$ martingale ($Y_t = y_t 1\!\!1_{t<\tau} + y_t(\tau)1\!\!1_{\tau\leq t}$) if and only if

  for any u, the processes $y_t(u), t \geq u$) are $\mathbb{F}$ martingales

  $\mathbb{E}(Y_t|\mathcal{F}_t) = y_t G_t + \int_t^\infty y_t(s) g_t(s)\nu(ds)$ is an $\mathbb{F}$ martingale

• If W is a $\mathbb{F}$-Brownian motion, then

$$W_t = \widetilde{W}_t + \int_0^{t\wedge\tau} \frac{d\langle x, G\rangle_s}{G_{s-}} + \int_{t\wedge\tau}^{t} \frac{d\langle x, g_{.}(\tau)\rangle_s}{g_{s-}(\tau)},$$

where W is a $\mathbb{G}$-Brownian motion.

Credit derivatives

1. Dynamics of corporate bonds
2. CDS
3. Hedging
4. Swaptions

A generic defaultable claim (X, A, Z, τ) consists of:

1. A promised contingent claim X representing the payoff received by the holder of the claim at time T, if no default has occurred prior to or at maturity date T.
2. A process A representing the dividends stream prior to default.
3. A recovery process Z representing the recovery payoff at time of default, if default occurs prior to or at maturity date T.
4. A random time τ representing the default time.

Dynamics of corporate bonds in a Cox Model

Let $B(t,T)$ be the price at time t of a default-free bond paying 1 at maturity T satisfies

$$B(t,T) = \mathbb{E}_{\mathbb{Q}}\Big(\exp\big(- \int_t^T r_s \, ds\big) \,\Big|\, \mathcal{F}_t \Big)$$

where $\mathbb{Q}$ is the risk-neutral probability.

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The market price $D(t,T)$ of a defaultable zero-coupon bond with maturity T is

$$\begin{aligned} D(t,T) &= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{T<\tau\}} \exp\big(-\int_t^T r_s\, ds\big) \,\Big|\, \mathcal{G}_t \Big) \\ &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}_{\mathbb{Q}}\Big(\exp\big(-\int_t^T [r_s + \lambda_s^{\mathbb{Q}}]\, ds\big) \,\Big|\, \mathcal{F}_t \Big). \end{aligned}$$

Promised payoff:

Let $X \in \mathcal{F}_T$

$$\mathbb{E}_{\mathbb{Q}}\left(X \mathbb{1}_{T<\tau} \exp - \int_t^T r_s ds | \mathcal{G}_t\right) = \mathbb{1}_{t<\tau} \mathbb{E}_{\mathbb{Q}}\left(X \exp - \int_t^T (r_s + \lambda_s) ds | \mathcal{F}_t\right)$$

$\lambda (= \lambda^{\mathbb{Q}})$ is also called the **spread**.

Recovery paid at Maturity

We consider a contract which pays Z_τ at date T, if $\tau \leq T$ where Z is an $\mathbb{F}$-adapted process and no payment in the case $\tau > T$. We also assume that the interest rate is null. The price at time t of this contract is

$$\begin{aligned} S_t &= \mathbb{E}_{\mathbb{Q}}(Z_\tau 1\!\!1_{\tau \leq T} | \mathcal{G}_t) \\ &= \int_0^t Z_u dH_u + L_t \left(- \int_0^t Z_u e^{-\Lambda_u} \lambda_u du + m_t^Z \right) \end{aligned}$$

where $m_t^Z = \mathbb{E}_{\mathbb{Q}}(\int_0^T Z_u e^{-\Lambda_u} \lambda_u du | \mathcal{F}_t)$ is an $\mathbb{F}$ (hence a $\mathbb{G}$) martingale and $L_t = 1\!\!1_{t < \tau} e^{\Lambda_t}$.

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We assume here that $\mathbb{F}$-martingales are continuous. From $dL_t = -L_{t-} dM_t$ and integration by parts formula we deduce that

$$dS_t = (Z_t - S_{t-})\, dM_t + L_t dm_t^Z$$

$dM_t = dH_t - (1 - H_t)\lambda_t dt$

Recovery paid at Default

If the payment Z is done at time τ

$$S_t \quad = \quad 1\!\!1_{t<\tau}\mathbb{E}_{\mathbb{Q}}(Z_\tau 1\!\!1_{t<\tau<T}|\mathcal{G}_t) = L_t\left(-\int_0^t Z_u e^{-\Lambda_u}\lambda_u du + m_t^Z\right)$$

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where $m_t^Z = \mathbb{E}_{\mathbb{Q}}(\int_0^T Z_u e^{-\Lambda_u}\lambda_u du|\mathcal{F}_t)$. Then,

$$\begin{aligned} dS_t &= -Z_t\lambda_t(1-H_t)dt - S_{t-}dM_t + L_t dm_t^Z \\ &= (Z_t - S_{t-})dM_t + L_t dm_t^Z - Z_t dH_t\,. \end{aligned}$$

The process $S_t + \int_0^t Z_s(1-H_s)\lambda_s ds$ is a $\mathbb{G}$ martingale, as well as $S_t + Z_\tau H_t$.

Price and Hedging a defaultable call

The savings account $Y^0_t = 1$, a risky asset with risk-neutral dynamics $dY_t = Y_t \sigma dW_t$ and a DZC of maturity T with price $D(t, T)$ are traded. The reference filtration is that of a Brownian motion W.

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$$D(t,T) = L_t \mathbb{Q}(\tau > T | \mathcal{F}_t) = L_t m_t$$

with $m_t = \mathbb{Q}(\tau > T | \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(e^{-\Lambda_T} | \mathcal{F}_t)$.

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The price of a defaultable call with payoff $1\!\!1_{T<\tau}(Y_T - K)^+$ is

$$C_t = \mathbb{E}_{\mathbb{Q}}(1\!\!1_{T<\tau}(Y_T - K)^+ | \mathcal{G}_t) = 1\!\!1_{t<\tau} e^{\Lambda_t} \mathbb{E}_{\mathbb{Q}}(e^{-\Lambda_T}(Y_T - K)^+ | \mathcal{F}_t)$$

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with $m_t^Y = \mathbb{E}_{\mathbb{Q}}(e^{-\Lambda_T}(Y_T - K)^+|\mathcal{F}_t)$, hence

$$dC_t = L_t dm_t^Y - m_t^Y L_{t-} dM_t = \frac{C_{t-}}{D(t,T)} dD(t,T) - L_t \frac{m_t^Y}{m_t} dm_t + L_t dm_t^Y$$

An hedging strategy consists of holding $\frac{C_{t-}}{D(t,T)}$ DZCs.

Credit Default Swap under (H) Hypothesis

Valuation of a Credit Default Swap

A CDS issued at time s, with maturity T, and recovery δ at default is a defaultable claim $(0, A, Z, \tau)$ where

$$dA_t = -\kappa \mathbb{1}_{]0,T]}(t)\, dt, \quad Z_t = \delta_t \mathbb{1}_{[0,T]}(t).$$

A credit default swap (CDS) is a contract between two counterparties. B agrees to pay a default payment Z to A if a default of the obligor C occurs. If there is no default until the maturity of the default swap, B pays nothing. A pays a fee for the default protection. The fee can be either a fee paid till the maturity or till the default event.

A can not cancelled the contract. He can at any time before the default transfer the contract to D: D will pay the fee and receive the default payment if any. As we shall see, it can happen that D will require an amount of cash to accept to receive the contract. Usually, the fee consists of C_i paid at time T_i (this is the fixed leg). However, here we shall consider a continuous payment. The default payment is called the default leg.

A stylized credit default swap is formally introduced through the following definition.

A ***credit default swap*** *with a* ***constant spread*** κ *and* ***recovery at default*** δ *is a contract:*

the buyer of protection pays a premium κdt in the time interval $[t, t + dt]$ up to $T \wedge \tau$

The seller pays a recovery $\delta(\tau)$ at time τ, in the case $\tau < T$.

Ex-dividend Price of a CDS

We now assume that **(H) hypothesis holds** between $\mathbb{F}$ and $\mathbb{G}$, that is $\mathbb{F}$-martingales are $\mathbb{G}$-martingales. Then, F is increasing. We assume that F is absolutely continuous w.r.t. Lebesgue measure. Then the process

$$M_t = H_t - \int_0^{t\wedge\tau} \lambda_u \, du,$$

with $\lambda_t dt = \frac{dF_t}{G_t}$ is a $\mathbb{G}$-martingale.

$G_t = \mathbb{Q}(\tau > t | \mathcal{F}_t) = 1 - F_t$

The ex-dividend price of a credit default swap, with a rate process κ and a protection payment δ_τ at default, equals, for every $t \in [0, T]$

$$\begin{aligned} S_t(\kappa) &= \mathbb{E}_{\mathbb{Q}}\Big(\delta(\tau)\mathbb{1}_{\{t<\tau\leq T\}} - \mathbb{1}_{\{t<\tau\}}\kappa\big((\tau \wedge T) - t\big) \,\Big|\, \mathcal{G}_t\Big) \\ &= \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}\Big(\int_t^T B_u^{-1} G_u(\delta_u \lambda_u - \kappa)\, du \,\Big|\, \mathcal{F}_t\Big) \end{aligned}$$

and thus the cumulative price of a CDS equals, for any $t \in [0, T]$,

$$S_t^{\text{cum}}(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}\Big(\int_t^T B_u^{-1} G_u(\delta_u \lambda_u - \kappa)\, du \,\Big|\, \mathcal{F}_t\Big) + B_t \int_{]0,t]} B_u^{-1}\, dD_u.$$

The dividend process $D(\kappa, \delta, T, \tau)$ of a CDS equals

$$D_t = \int_{]0,t\wedge T]} \delta_u \, dH_u - \kappa \int_{]0,t\wedge T]} (1 - H_u)\, du = \delta_\tau \mathbb{1}_{\{\tau \leq t\}} - \kappa(t \wedge T \wedge \tau).$$

$B_t = \exp(\int_0^t r_s ds)$

Trading Strategies with a CDS

A strategy $\phi_t = (\phi_t^0, \phi_t^1)$, $t \in [0, T]$ is **self-financing** if the wealth process $U(\phi)$, defined as

$$U_t(\phi) = \phi_t^0 + \phi_t^1 S_t(\kappa),$$

satisfies

$$dU_t(\phi) = \phi_t^1\, dS_t(\kappa) + \phi_t^1\, dD_t,$$

where $S(\kappa)$ is the ex-dividend price of a CDS with the dividend stream D. A strategy ϕ replicates a contingent claim Y if $U_T(\phi) = Y$.

Hedging of a Contingent Claim in the CDS Market

We assume that $\mathbb{F}$ is the trivial filtration. Our aim is to find **a replicating strategy for the defaultable claim** $(X, 0, Z, \tau)$, where X is a constant and $Z_t = z(t)$. Let

$$Y = \mathbb{1}_{\{T \geq \tau\}} z(\tau) + \mathbb{1}_{\{T < \tau\}} X$$

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Let $\widetilde{y}$ and ϕ^1 be defined as

$$\widetilde{y}(t) = \frac{1}{G(t)} \left(XG(T) - \int_t^T z(s) dG(s) \right)$$

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Let $\phi_t^0 = V_t(\phi) - \phi^1(t) S_t(\kappa)$, where $V_t(\phi) = \mathbb{E}_{\mathbb{Q}}(Y | \mathcal{H}_t)$

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Let $\phi_t^0 = V_t(\phi) - \phi^1(t) S_t(\kappa)$, where $V_t(\phi) = \mathbb{E}_{\mathbb{Q}}(Y | \mathcal{H}_t)$. Then the self-financing strategy $\phi = (\phi^0, \phi^1)$ based on the savings account and the CDS is a replicating strategy.

Proof: The terminal value of the wealth is

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On the one hand

$$\begin{aligned}
\mathbb{E}(Y|\mathcal{H}_t) = Y_t &= z(\tau)\mathbb{1}_{\{\tau\le t\}} + \mathbb{1}_{\{t<\tau\}}\frac{1}{G(t)}\left(XG(T) - \int_t^T z(s)dG(s)\right) \\
&= \int_0^t z(s)dH_s + (1-H_t)\frac{1}{G(t)}\left(XG(T) - \int_t^T z(s)dG(s)\right)
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hence $dY_t = (z(t) - \widetilde{y}(t))\,dM_t$ with $\widetilde{y}(t) = \frac{1}{G(t)}(XG(T) - \int_t^T z(s)dG(s))$.

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hence $dY_t = (z(t) - \widetilde{y}(t))\, dM_t$ with $\widetilde{y}(t) = \frac{1}{G(t)}(XG(T) - \int_t^T z(s)dG(s))$.
On the other hand,
$dY_t = \phi_t^1\,(dS_t(\kappa) - \kappa(1-H_t)dt + \delta(t)dH_t) = \phi_t^1(\delta(t) - S_{t-}(\kappa))\, dM_t$.

Hedging of credit derivatives

1. Two default free assets, one defaultable asset
    1.1 Two default free assets, one total default asset
    1.2 Two default free assets, one defaultable with recovery
2. Two defaultable assets

Two default-free assets, one defaultable asset

We present a particular case where there are two default-free assets

- the savings account Y^1 with constant interest rate r

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- a defaultable asset

$$dY_t^3 = Y_{t-}^3(\mu_{3,t}dt + \sigma_{3,t}dW_t + \kappa_{3,t}dM_t)\,,$$

where the coefficients $\mu_3, \sigma_3, \kappa_3$ are $\mathbb{G}$-adapted processes with $\kappa_3 \geq -1$.

Here, M is the compensated martingale of the default process

$$M_t = H_t - \int_0^t (1 - H_s)\lambda_s ds$$

W is an $\mathbb{F}$ and a $\mathbb{G}$-Brownian motion, where $\mathbb{F}$ is the natural filtration of W and $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, λ is an $\mathbb{F}$ adapted process.

Our aim is to hedge defaultable claims. As we shall establish, the case of **total default** for the third asset (i.e. $\kappa_{3,t} \equiv -1$) is really different from the others.

Two default-free assets, a total default asset

We assume that

$$dY^3_t = Y^3_{t-}\left(\mu_{3,t}dt + \sigma_{3,t}dW_t - dM_t\right).$$

$$Y^3_t = \widetilde{Y}^3_t \mathbb{1}_{t<\tau}\,.$$

Arbitrage condition, completeness of the market

Our aim is to determine the emm(s) $\mathbb{Q}$ for the model

$$\begin{aligned} dY_t^1 &= Y_t^1 r dt \\ dY_t^2 &= Y_t^2(\mu_2 dt + \sigma_2 dW_t) \\ dY_t^3 &= Y_{t-}^3(\mu_3 dt + \sigma_3 dW_t - dM_t) \,. \end{aligned}$$

when Y^1 is the numéraire. The probability $\mathbb{Q}$ such that $Y^{i,1} = Y^i/Y^1$ are martingales is

$$d\mathbb{Q}|_{\mathcal{G}_t} = L_t d\mathbb{P}|_{\mathcal{G}_t} \,,$$

where

$$dL_t = L_{t-}(\theta_t dW_t + \zeta_t dM_t)$$

with

$$\begin{aligned}\theta &= \frac{r-\mu}{\sigma_2}\\ \zeta\lambda &= \mu_3 - r + \sigma_3\,\frac{r-\mu}{\sigma_2}\,,\end{aligned}$$

as soon as $\zeta > -1$.

Under $\mathbb{Q}$

$$
\begin{aligned}
W_t^* &= W_t - \int_0^t \theta_s ds \\
M_t^* &= M_t - \int_0^t (1 - H_s)\lambda_s \zeta_s ds
\end{aligned}
$$

are martingales.

Under $\mathbb{Q}$, the process

$$H_t - \int_0^t (1 - H_s)\lambda_s^* ds$$

is a martingale where

$$\lambda_t^* = \lambda_t(1 + \zeta_t)$$

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The hedging strategy consists of a triple of predictable processes ϕ^1, ϕ^1, ϕ^3 such that

$$\phi^3_t Y^3_t = C_t, \forall t < \tau, \quad \phi^1_t e^{rt} + \phi^2_t Y^2_t = 0$$

and which satisfies the self financing condition

$$dC_t = \phi^1_t r e^{rt} dt + \phi^2_t dY^2_t + \phi^3_t dY^3_t$$

$dY^1 = Y^1 r dt,\ dY^2 = Y^2(\mu_2 dt + \sigma_2 dW), dY^3 = Y^3(\mu_3 dt + \sigma_3 dW - dM)$

Indeed, under $\mathbb{Q}$, for $r = 0$, one has

$$dC_t = \alpha_t dW_t^* - C_{t-} dM_t^*$$

and,

$$\phi_t^2 dY_t^2 + \phi_t^3 dY_t^3 \quad = \quad \phi_t^2 \sigma_2 Y_t^2 dW_t^* - C_{t-} dM_t^*$$

hence the existence of ϕ^2, and ϕ^1 such that $\phi_t^1 e^{rt} + \phi_t^2 Y_t^2 = 0$

PDE Approach

We are working in a model with constant (or Markovian) coefficients

$$
\begin{aligned}
dY_t &= Y_t r dt \\
dY_t^2 &= Y_t^2(\mu_2 dt + \sigma_2 dW_t) \\
dY_t^3 &= Y_{t-}^3(\mu_3 dt + \sigma_3 dW_t - dM_t)\,.
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\end{aligned}
$$

Let $C(t, Y_t^2, Y_t^3, H_t)$ be the price of the contingent claim $G(Y_T^2, Y_T^3, H_T)$ and λ^* be the risk-neutral intensity of default.

Then,

$$\partial_t C(t, y_2, y_3; 0) + r y_2 \partial_2 C(t, y_2, y_3; 0) + r^* y_3 \partial_3 C(t, y_2, y_3; 0) - r^* C(t, y_2, y_3; 0)$$
$$+ \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3; 0) + \lambda^* C(t, y_2, 0; 1) = 0$$

where $r^* = r + \lambda^*$

Then,

$$\partial_t C(t,y_2,y_3;0) + ry_2\partial_2 C(t,y_2,y_3;0) + r^* y_3 \partial_3 C(t,y_2,y_3;0) - r^* C(t,y_2,y_3;0) + \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} C(t,y_2,y_3;0) + \lambda^* C(t,y_2,0;1) = 0$$

where $r^* = r + \lambda^*$ and

$$\partial_t C(t,y_2;1) + ry_2\partial_2 C(t,y_2;1) + \frac{1}{2}\sigma_2^2 y_2^2 \partial_{22} C(t,y_2;1) - rC(t,y_2;1) = 0$$

Then,

$$\partial_t C(t, y_2, y_3; 0) + ry_2 \partial_2 C(t, y_2, y_3; 0) + r^* y_3 \partial_3 C(t, y_2, y_3; 0) - r^* C(t, y_2, y_3; 0) + \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3; 0) + \lambda^* C(t, y_2, 0; 1) = 0$$

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with the terminal conditions

$$C(T, y_2, y_3; 0) = G(y_2, y_3; 0), \quad C(T, y_2; 1) = G(y_2, 0; 1).$$

The ***replicating strategy*** ϕ *for* Y *is given by formulae*

$$\begin{aligned}
\phi_t^3 Y_{t-}^3 &= -\Delta C(t) = -C(t, Y_t^2, 0; 1) + C(t, Y_t^2, Y_{t-}^3; 0) \\
\sigma_2 \phi_t^2 Y_t^2 &= -\Delta C(t) + \sum_{i=2}^{3} Y_{t-}^i \sigma_i \partial_i C(t) \\
\phi_t^1 Y_t^1 &= C(t) - \phi_t^2 Y_t^2 - \phi_t^3 Y_t^3 \,.
\end{aligned}$$

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$$\begin{aligned}
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\sigma_2 \phi^2_t Y^2_t &= -\Delta C(t) + \sum_{i=2}^{3} Y^i_{t-} \sigma_i \partial_i C(t) \\
\phi^1_t Y^1_t &= C(t) - \phi^2_t Y^2_t - \phi^3_t Y^3_t \, .
\end{aligned}$$

Note that, in the case of survival claim, $C(t, Y^2_t, 0; 1) = 0$ and $\phi^3_t Y^3_{t-} = C(t, Y^2_{t-}, Y^3_{t-}; 0)$ for every $t \in [0, T]$.

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$$\phi^3_t Y^3_t = C(t, Y^2_t, Y^3_t; 0), \quad \phi^1_t Y^1_t + \phi^2_t Y^2_t = 0.$$

The last equality is a special case of the **balance condition**. It ensures that the wealth of a replicating portfolio falls to 0 at default time.

Example 1

Consider a survival claim $Y = \mathbb{1}_{\{T<\tau\}} g(Y_T^2)$. Its pre-default pricing function $C(t, y_2, y_3\,; 0) = C^g(t, y_2)$ where C^g solves

$$\begin{aligned} \partial_t C^g(t,y;0) + ry\partial_2 C^g(t,y;0) + \frac{1}{2}\sigma_2^2 y^2 \partial_{22} C^g(t,y;0) - r^* C^g(t,y;0) &= 0 \\ C^g(T,y;0) &= g(y) \end{aligned}$$

The solution is

$$C^g(t,y) = e^{(r^*-r)(t-T)}\, C^{r,g,2}(t,y) = e^{\lambda^*(t-T)}\, C^{\,r,g,2}(t,y),$$

where $C^{\,r,g,2}$ is the price of an option with payoff $g(Y_T)$ in a BS model with interest rate r and volatility σ_2.

Example 2

Consider a survival claim of the form

$$Y = G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T<\tau\}} g(Y_T^3).$$

Then the post-default pricing function $C^g(\cdot\,;1)$ vanishes identically, and the pre-default pricing function $C^g(\cdot\,;0)$ is

$$C^g(t, y_2, y_3; 0) = C^{r^*,g,3}(t, y_3)$$

where $C^{\alpha,g,3}(t,y)$ is the price of the contingent claim $g(Y_T)$ in the Black-Scholes framework with the interest rate α and the volatility parameter equal to σ_3.

Two default-free assets, one defaultable asset with Recovery, PDE approach

Let the price processes Y^1, Y^2, Y^3 *satisfy*

$$\begin{aligned} dY_t^1 &= rY_t^1 dt \\ dY_t^2 &= Y_t^2(\mu_2 dt + \sigma_2 dW_t) \\ dY_t^3 &= Y_{t-}^3(\mu_3 dt + \sigma_3 dW_t + \kappa_3 dM_t) \end{aligned}$$

with $\sigma_2 \neq 0$ *and where the coefficients are constant. Assume that the relationship* $\sigma_2(r - \mu_3) = \sigma_3(r - \mu_2)$ *holds and* $\kappa_3 \neq 0, \kappa_3 > -1$.

Two default-free assets, one defaultable asset with Recovery, PDE approach

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with $\sigma_2 \neq 0$ *and where the coefficients are constant. Assume that the relationship* $\sigma_2(r-\mu_3) = \sigma_3(r-\mu_2)$ *holds and* $\kappa_3 \neq 0, \kappa_3 > -1$. *Then the price of a contingent claim* $Y = G(Y_T^2, Y_T^3, H_T)$ *can be represented as* $\pi_t(Y) = C(t, Y_t^2, Y_t^3; H_t)$, *where the pricing functions* $C(\cdot\,; 0)$ *and* $C(\cdot\,; 1)$ *satisfy the following PDEs*

$$\partial_t C(t, y_2, y_3; 1) + r y_2 \partial_2 C(t, y_2, y_3; 1) + r y_3 \partial_3 C(t, y_2, y_3; 1) - r C(t, y_2, y_3; 1) + \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3; 1) = 0$$

$$\partial_t C(t, y_2, y_3; 1) + r y_2 \partial_2 C(t, y_2, y_3; 1) + r y_3 \partial_3 C(t, y_2, y_3; 1) - r C(t, y_2, y_3; 1)$$
$$+ \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3; 1) = 0$$

and

$$\begin{aligned} &\partial_t C(t, y_2, y_3; 0) + r y_2 \partial_2 C(t, y_2, y_3; 0) + y_3 \left(r - \kappa_3 \lambda\right) \partial_3 C(t, y_2, y_3; 0) \\ - \quad & r C(t, y_2, y_3; 0) + \frac{1}{2} \sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t, y_2, y_3; 0) \\ & + \lambda \Big(C(t, y_2, y_3(1 + \kappa_3); 1) - C(t, y_2, y_3; 0) \Big) = 0 \end{aligned}$$

$$\partial_t C(t,y_2,y_3;1) + ry_2\partial_2 C(t,y_2,y_3;1) + ry_3\partial_3 C(t,y_2,y_3;1) - rC(t,y_2,y_3;1) + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} C(t,y_2,y_3;1) = 0$$

and

$$\begin{aligned}\partial_t C(t,y_2,y_3;0) + ry_2\partial_2 C(t,y_2,y_3;0) + y_3\left(r-\kappa_3\lambda\right)\partial_3 C(t,y_2,y_3;0) \\ - \quad rC(t,y_2,y_3;0) + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} C(t,y_2,y_3;0) \\ +\lambda\Big(C(t,y_2,y_3(1+\kappa_3);1) - C(t,y_2,y_3;0)\Big) = 0\end{aligned}$$

subject to the terminal conditions

$$C(T,y_2,y_3;0) = G(y_2,y_3,0), \quad C(T,y_2,y_3;1) = G(y_2,y_3,1).$$

The replicating strategy equals $\phi = (\phi^1, \phi^2, \phi^3)$

$$\begin{aligned}
\phi_t^2 &= \frac{1}{\sigma_2 \kappa_3 Y_t^2} \left(\kappa_3 \sum_{i=2}^{3} \sigma_i y_i \partial_i C(t, Y_t^2, Y_{t-}^3, H_{t-}) \right. \\
&\quad \left. - \sigma_3 \big(C(t, Y_t^2, Y_{t-}^3(1+\kappa_3); 1) - C(t, Y_t^2, Y_{t-}^3; 0) \big) \right), \\
\phi_t^3 &= \frac{1}{\kappa_3 Y_{t-}^3} \big(C(t, Y_t^2, Y_{t-}^3(1+\kappa_3); 1) - C(t, Y_t^2, Y_{t-}^3; 0) \big),
\end{aligned}$$

and where ϕ_t^1 *is given by* $\phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3 = C_t$.

Example Consider a survival claim of the form

$$Y = G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T<\tau\}} g(Y_T^3).$$

Then the post-default pricing function $C^g(\cdot\,;1)$ vanishes identically, and the pre-default pricing function $C^g(\cdot\,;0)$ solves

$$\begin{aligned}\partial_t C^g(\cdot\,;0) &+ r y_2 \partial_2 C^g(\cdot\,;0) + y_3\left(r-\kappa_3\lambda\right)\partial_3 C^g(\cdot\,;0)\\ &+ \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} C^g(\cdot\,;0) - (r+\lambda) C^g(\cdot\,;0) = 0\\ C^g(T, y_2, y_3; 0) &= g(y_3)\end{aligned}$$

Denote $\alpha = r - \kappa_3\lambda$ and $\beta = \lambda(1+\kappa_3)$.

Then, $C^g(t, y_2, y_3; 0) = e^{\beta(T-t)} C^{\alpha,g,3}(t, y_3)$ where $C^{\alpha,g,3}(t,y)$ is the price of the contingent claim $g(Y_T)$ in the Black-Scholes framework with the interest rate α and the volatility parameter equal to σ_3.

Let C_t be the current value of the contingent claim Y, so that

$$C_t = \mathbb{1}_{\{t<\tau\}} e^{\beta(T-t)} C^{\alpha,g,3}(t, y_3).$$

The hedging strategy of the survival claim is, on the event $\{t < \tau\}$,

$$\begin{aligned}\phi_t^3 Y_t^3 &= -\frac{1}{\kappa_3} e^{-\beta(T-t)} C^{\alpha,g,3}(t, Y_t^3) = -\frac{1}{\kappa_3} C_t,\\ \phi_t^2 Y_t^2 &= \frac{\sigma_3}{\sigma_2}\left(Y_t^3 e^{-\beta(T-t)} \partial_y C^{\alpha,g,3}(t, Y_t^3) - \phi_t^3 Y_t^3\right).\end{aligned}$$

Hedging of a Recovery Payoff

The price C^g of the payoff $G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T\geq\tau\}} g(Y_T^2)$ solves

$$\begin{aligned} \partial_t C^g(\cdot\,;1) + ry\partial_y C^g(\cdot\,;1) + \frac{1}{2}\sigma_2^2 y^2 \partial_{yy} C^g(\cdot\,;1) - rC^g(\cdot\,;1) &= 0 \\ C^g(T,y;1) &= g(y) \end{aligned}$$

hence $C^g(t, y_2, y_3, 1) = C^{r,g,2}(t, y_2)$ is the price of $g(Y_T^2)$ in the model Y^1, Y^2. Prior to default, the price of the claim solves

$$\begin{aligned} \partial_t C^g(\cdot;0) &+ ry_2\partial_2 C^g(\cdot\,;0) + y_3\left(r - \kappa_3\lambda\right)\partial_3 C^g(\cdot\,;0) \\ &+ \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i\sigma_j y_i y_j \partial_{ij} C^g(\cdot\,;0) - (r+\lambda)C^g(\cdot\,;0) = -\lambda C^g(t, y_2; 1) \\ C^g(T, y_2, y_3; 0) &= 0 \end{aligned}$$

Hence $\qquad C^g(t, y_2, y_3; 0) = (1 - e^{\lambda(t-T)})C^{r,g,2}(t, y_2).$

Two defaultable assets with total default

Assume that Y^1 and Y^2 are defaultable tradeable assets with zero recovery and a common default time τ.

$$dY_t^i = Y_{t-}^i(\mu_i dt + \sigma_i dW_t - dM_t), i = 1, 2$$

Then

$$Y_t^1 = 1\!\!1_{\{\tau > t\}} \widetilde{Y}_t^1, \quad Y_t^2 = 1\!\!1_{\{\tau > t\}} \widetilde{Y}_t^2$$

with

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i((\mu_i + \lambda_t)dt + \sigma_i dW_t), i = 1, 2$$

The wealth process V associated with the self-financing trading strategy (ϕ^1, ϕ^2) satisfies for $t \in [0, T]$

$$V_t = Y_t^1 \left(V_0^1 + \int_0^t \phi_u^2 \, d\widetilde{Y}_u^{2,1} \right)$$

where $\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^2 / \widetilde{Y}_t^1$.

Obviously, this market is **incomplete, however, some contingent claims are hedgeable**, as we present now.

$dY_t^i = Y_{t-}^i(\mu_i dt + \sigma_i dW_t - dM_t), i = 1, 2$

Hedging Survival claim: martingale approach

A strategy (ϕ^1, ϕ^2) replicates a survival claim $C = X \mathbb{1}_{\{\tau > T\}}$ whenever we have

$$\widetilde{Y}^1_T \Big(\widetilde{V}^1_0 + \int_0^T \phi^2_t \, d\widetilde{Y}^{2,1}_t \Big) = X$$

for some constant $\widetilde{V}^1_0$ and some $\mathbf{F}$-predictable process ϕ^2.

It follows that if $\sigma_1 \neq \sigma_2$, **any survival claim $C = X \mathbb{1}_{\{\tau > T\}}$ is attainable.**

Let $\widetilde{\mathbb{Q}}$ be a probability measure such that $\widetilde{Y}^{2,1}_t$ is an $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$. Then the pre-default value $\widetilde{U}_t(C)$ at time t of $(X, 0, \tau)$ equals

$$\widetilde{U}_t(C) = \widetilde{Y}^1_t \, \mathbb{E}_{\widetilde{\mathbb{Q}}} \Big(X (\widetilde{Y}^1_T)^{-1} \,|\, \mathcal{F}_t \Big).$$

Example: Call option on a defaultable asset We assume that $Y_t^1 = D(t,T)$ represents a defaultable ZC-bond with zero recovery, and $Y_t^2 = \mathbb{1}_{\{t<\tau\}}\widetilde{Y}_t^2$ is a generic defaultable asset with total default. The payoff of a call option written on the defaultable asset Y^2 equals

$$C_T = (Y_T^2 - K)^+ = \mathbb{1}_{\{T<\tau\}}(\widetilde{Y}_T^2 - K)^+$$

The replication of the option reduces to finding a constant x and an $\mathbb{F}$-predictable process ϕ^2 that satisfy

$$x + \int_0^T \phi_t^2 \, d\widetilde{Y}_t^{2,1} = (\widetilde{Y}_T^2 - K)^+.$$

Assume that the volatility $\sigma_{1,t} - \sigma_{2,t}$ of $\widetilde{Y}^{2,1}$ is deterministic. Let $\widetilde{F}_2(t,T) = \widetilde{Y}_t^2(\widetilde{D}(t,T))^{-1}$

The credit-risk-adjusted forward price of the option written on Y^2 equals

$$\widetilde{C}_t = \widetilde{Y}_t^2 \mathcal{N}\big(d_+(\widetilde{F}_2(t,T),t,T)\big) - K\widetilde{D}(t,T)\mathcal{N}\big(d_-(\widetilde{F}_2(t,T),t,T)\big),$$

where

$$d_\pm(\widetilde{f},t,T) = \frac{\ln \widetilde{f} - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma_{1,u} - \sigma_{2,u})^2 \, du.$$

Moreover the replicating strategy ϕ in the spot market satisfies for every $t \in [0,T]$, on the set $\{t < \tau\}$,

$$\phi_t^1 = -K\mathcal{N}\big(d_-(\widetilde{F}_2(t,T),t,T)\big), \quad \phi_t^2 = \mathcal{N}\big(d_+(\widetilde{F}_2(t,T),t,T)\big).$$

Hedging Survival claim: PDE approach

Assume that $\sigma_1 \neq \sigma_2$. Then the pre-default pricing function v satisfies the PDE

$$\partial_t C + y_1 \left(\mu_1 + \lambda - \sigma_1 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} \right) \partial_1 C + y_2 \left(\mu_2 + \lambda - \sigma_2 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} \right) \partial_2 C$$
$$+ \frac{1}{2} \Big(y_1^2 \sigma_1^2 \partial_{11} C + y_2^2 \sigma_2^2 \partial_{22} C + 2 y_1 y_2 \sigma_1 \sigma_2 \partial_{12} C \Big) = \left(\mu_1 + \lambda - \sigma_1 \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} \right) C$$

with the terminal condition $C(T, y_1, y_2) = G(y_1, y_2)$.

Hedging defaultable claims with CDSs

Our aim is to hedge

$$Y = \mathbb{1}_{\{T \geq \tau\}} Z_\tau + \mathbb{1}_{\{T < \tau\}} X.$$

using two CDS with maturities T_i, rates κ_i and protection payment δ^i. We assume $r = 0$. Let ζ^i_t defined as

$$m^i_t = \mathbb{E}_{\mathbb{Q}} \left(\int_0^T \delta^i_u G_u \lambda_u \, du - \kappa_i \int_0^T G_u \, du \,\bigg|\, \mathcal{F}_t \right) \, , \; dm^i_t = \zeta^i_t dW_t$$

and

$$m^Z_t = \mathbb{E}_{\mathbb{Q}}(- \int_0^\infty Z_u dG_u + G_T X | \mathcal{F}_t), \; dm^Z_t = \zeta^Z_t dW_t$$

Assume that there exist $\mathbb{F}$-predictable processes ϕ^1, ϕ^2 such that

$$\sum_{i=1}^{2} \phi_t^i \big(\delta_t^i - \widetilde{S}_t^i(\kappa_i)\big) = Z_t - \widetilde{y}_t, \quad \sum_{i=1}^{2} \phi_t^i \zeta_t^i = \zeta_t,$$

where $\widetilde{y}$ is given by

$$\widetilde{y}_t = \frac{1}{G_t}\, \mathbb{E}_{\mathbb{Q}} \left(- \int_t^T Z_u \, dG_u + G_T X \,\Big|\, \mathcal{F}_t \right).$$

Let $\phi_t^0 = V_t(\phi) - \sum_{i=1}^{2} \phi_t^i S_t^i(\kappa_i)$, where the process $V(\phi)$ is given by

$$dV_t(\phi) = \sum_{i=1}^{2} \phi_t^i \big(dS_t^i(\kappa_i) + dD_t^i \big)$$

with the initial condition $V_0(\phi) = \mathbb{E}_{\mathbb{Q}}(Y)$. Then the self-financing trading strategy $\phi = (\phi^0, \phi^1, \phi^2)$ is admissible and is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$.

Valuation of Credit Default Swaptions

A forward CDS issued at time s, with starting date U, maturity T, and recovery δ at default is a defaultable claim $(0, A, Z, \tau)$ where

$$dA_t = -\kappa \mathbb{1}_{]U,T]}(t)\, dt, \quad Z_t = \delta_t \mathbb{1}_{[U,T]}(t).$$

- The CDS rate κ is $\mathcal{F}_s$-measurable.
- The $\mathbb{F}$-adapted process $\delta : [U, T] \to \mathbb{R}$ represents the default protection.

The value of the forward CDS equals, for every $t \in [s, U]$,

$$S_t(\kappa) = B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{U<\tau\le T\}} B_\tau^{-1} \delta_\tau \,\Big|\, \mathcal{G}_t \Big) - \kappa\, B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\int_{]t\wedge U, \tau\wedge T]} B_u^{-1}\, du \,\Big|\, \mathcal{G}_t \Big).$$

Valuation of a Forward CDS

The value of a credit default swap started at s, equals, for every $t \in [s, U]$,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}} \left(-\int_U^T B_u^{-1} \delta_u \, dG_u - \kappa \int_{]U,T]} B_u^{-1} G_u \, du \,\middle|\, \mathcal{F}_t \right).$$

Note that $S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa)$ where the $\mathbb{F}$-adapted process $\widetilde{S}(\kappa)$ is the pre-default value. Moreover

$$\widetilde{S}_t(\kappa) = \widetilde{P}(t, U, T) - \kappa \, \widetilde{A}(t, U, T)$$

where

- $\widetilde{P}(t, U, T)$ is the pre-default value of the protection leg,
- $\widetilde{A}(t, U, T)$ is the pre-default value of the fee leg per one unit of κ.

Credit Default Swaption

A credit default swaption is a call option with expiry date $R \leq U$ and zero strike written on the value of the forward CDS issued at time $0 \leq s < R$, with start date U, maturity T, and an $\mathcal{F}_s$-measurable rate κ.

The swaption's payoff C_R at expiry equals $C_R = (S_R(\kappa))^+$.

For a forward CDS with an $\mathcal{F}_s$-measurable rate κ we have, for every $t \in [s, U]$,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{A}(t, U, T)(\kappa(t, U, T) - \kappa).$$

It is clear that

$$C_R = \mathbb{1}_{\{R<\tau\}} \widetilde{A}(R,U,T)(\kappa(R,U,T) - \kappa)^+.$$

A credit default swaption is formally equivalent to a call option on the forward CDS rate with strike κ. This option is knocked out if default occurs prior to R.

The price at time $t \in [s, R]$ of a credit default swaption equals

$$C_t = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}} \left(\frac{G_R}{B_R} \widetilde{A}(R,U,T)(\kappa(R,U,T) - \kappa)^+ \,\Big|\, \mathcal{F}_t \right).$$

Define an equivalent probability measure $\widehat{\mathbb{Q}}$ on $(\Omega, \mathcal{F}_R)$ by setting

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}} = \frac{M_R^A}{M_0^A}, \quad \mathbb{Q}\text{-a.s.}$$

where the $(\mathbb{Q}, \mathbb{F})$-martingale M^A is given by

$$M_t^A = \mathbb{E}_{\mathbb{Q}} \Big(\int_{]U,T]} B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_t \Big).$$

The price of the credit default swaption equals, for every $t \in [s, R]$,

$$C_t = \mathbb{1}_{\{t<\tau\}} \widetilde{A}(t,U,T) \, \mathbb{E}_{\widehat{\mathbb{Q}}} \big((\kappa(R,U,T) - \kappa)^+ \,\big|\, \mathcal{F}_t \big) = \mathbb{1}_{\{t<\tau\}} \widetilde{C}_t.$$

The forward CDS rate $(\kappa(t,U,T),\ t \le R)$ is a $(\widehat{\mathbb{Q}}, \mathbb{F})$-martingale.

Brownian Case

- Let the filtration $\mathbb{F}$ be generated by a Brownian motion W under $\mathbb{Q}^*$.
- Since $M^P t = -\mathbb{E}_{\mathbb{Q}}\Big(\int_U^T B_u^{-1} \delta_u \, dG_u \,\Big|\, \mathcal{F}_t \Big)$ and M^A are strictly positive $(\mathbb{Q}, \mathbb{F})$-martingales, we have that

$$dM_t^P = M_t^P \sigma_t^P \, dW_t, \quad dM_t^A = M_t^A \sigma_t^A \, dW_t,$$

for some $\mathbb{F}$-adapted processes σ^P and σ^A.

The forward CDS rate $(\kappa(t, U, T),\, t \in [0, R])$ is $(\widehat{\mathbb{Q}}, \mathbb{F})$-martingale and

$$d\kappa(t, U, T) = \kappa(t, U, T)\sigma_t^\kappa \, d\widehat{W}_t$$

where $\sigma^\kappa = \sigma^P - \sigma^A$ and the $(\widehat{\mathbb{Q}}, \mathbb{F})$-Brownian motion $\widehat{W}$ equals

$$\widehat{W}_t = W_t - \int_0^t \sigma_u^A \, du, \quad \forall\, t \in [0, R].$$

Assume that the volatility $\sigma^\kappa = \sigma^P - \sigma^A$ of the forward CDS spread is deterministic. Then the pre-default value of the credit default swaption with strike level κ and expiry date R equals, for every $t \in [0, U]$,

$$\widetilde{C}_t = \widetilde{A}_t \Big(\kappa_t \, N\big(d_+(\kappa_t, U-t)\big) - \kappa \, N\big(d_-(\kappa_t, U-t)\big)\Big)$$

where $\kappa_t = \kappa(t, U, T)$ and $\widetilde{A}_t = \widetilde{A}(t, U, T)$. Equivalently,

$$\widetilde{C}_t = \widetilde{P}_t \, N\big(d_+(\kappa_t, t, R)\big) - \kappa \, \widetilde{A}_t \, N\big(d_-(\kappa_t, t, R)\big)$$

where $\widetilde{P}_t = \widetilde{P}(t, U, T)$ and

$$d_\pm(\kappa_t, t, R) = \frac{\log(\kappa_t/\kappa) \pm \frac{1}{2}\int_t^R (\sigma^\kappa(u))^2 \, du}{\sqrt{\int_t^R (\sigma^\kappa(u))^2 \, du}} \, .$$